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Integration

Once you have a derivative, the natural question is "how do we integrate?"
An important part of the answer to this is the measure of integration, i.e. with respect to what are we integrating? There are many possibilities, but one is "natural."

So far we have been looking only at the components of forms. Here it will be useful to remember that the components arise when we decompose the form onto a basis.

What is the basis? From the "proto-vector" $\vec{\Delta s} = \Delta x^\mu e_{(\mu)}$ and dual definition $e^{(\mu)} e_{(\nu)} = \delta^\mu_\nu$ we find that a coordinate adapted basis of dual vectors are coordinate differential one-forms. That is:

$A^{(p)} = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_p}$ where the differentials anticommute, i.e. $dx^\mu dx^\nu = -dx^\nu dx^\mu$

Consider $A^{(2)} = \frac{1}{2} A_{\mu\nu} dx^\mu dx^\nu$ but since μ, ν are dummy we can rename them $\mu \leftrightarrow \nu$

$= \frac{1}{2} \underline{A_{\nu\mu}}\, \underline{dx^\nu dx^\mu}$

but $= -A_{\mu\nu}$ so $= -dx^\mu dx^\nu$

In fact we could consider this as $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$.

What is so nice about a set of anticommuting differentials?

Consider $dx\,dy$ and transform to $x'(x,y)$ $y'(x,y)$. Then:

$$dx\,dy \longrightarrow dx'dy' = \left(\frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy\right)\left(\frac{\partial y'}{\partial x} dx + \frac{\partial y'}{\partial y} dy\right) = \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} dx\,dx + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} dy\,dy + \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} dx\,dy + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x} dy\,dx$$

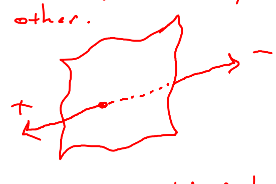
However, we know how $dx\,dy$ should transform... w/ the Jacobian, i.e. $dx'dy' = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right) dx\,dy$

Which is exactly what we get if $dx\,dy = -dy\,dx \Rightarrow dx\,dx = dy\,dy = 0$

So at the end of the day, the basis of a p-form is actually an integration measure over a p-dimensional (oriented) volume!

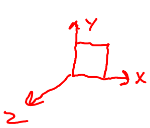
Where does orientation come from? Orientability is the possibility of "signing" the "area" $(+,-)$ to distinguish one "side" from the other.

Clearly this cannot be done on an unoriented surface like the Mobius strip.



If we break the surface up into infinitesimal pieces, e.g. w/ area $|dx\,dy|$ the question is how do we assign a "direction" to this.

In less sophisticated approaches we might set up



and say $d\vec{a} = \hat{n}\,da$ ← $dx\,dy$
$\llcorner (0,0,\pm 1)$

But this breaks in $d \neq 3$!

To overcome it, we forget the right hand rule:

so just use this!

Consider: dy □ dx and the right hand rule: $\left.\begin{array}{l} dx\,dy \ \odot \\ dy\,dx \ \otimes \end{array}\right\}$ or $\begin{array}{l} dx\,dy \ + \\ dy\,dx \ - \end{array}$ $\Rightarrow$ $\overbrace{dx\,dy = -dy\,dx}$

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In the end an expression like $\int_{\Sigma_p} A^{(p)}$ is perfectly well defined and coordinate invariant.

The physical significance of this is that there is a "natural" coupling between p-form fields $A^{(p)}$ and the p-dimensional "world-surfaces" swept out by (p-1)-dimensional objects.

Example: $p=1$ $\int_{\Sigma_1} A^{(1)}$ is the natural coupling of a 1-form A_μ to a particle's worldline.

Example: $p=2$ $\int_{\Sigma_2} B^{(2)}$ is the natural coupling of a 2-form $B_{\mu\nu}$ to a string's worldsheet.

In fact $\int_{\Sigma_p} B^{(p)}$ is the natural coupling of a p-form to a D(p-1)-brane's worldvolume

To see the mathematical power of forms consider the form version of Stoke's theorem:

For an orientable p-dimensional manifold Σ we have $\int_{\partial\Sigma} A^{(p-1)} = \int_{\Sigma} dA^{(p-1)}$

We can take this, and by considering special cases of dimension and dualization arrive at the $\underbrace{\text{Kelvin-Stokes theorem}}_{\text{this is the version for vector fields that you are familiar with}}$, the Divergence theorem, Green's theorem and the fundamental theorem of calculus!

Wednesday, January 31, 2018
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Action Principle in Classical Physics

Motivated by our examples from last time (largely focussed on actions for freely propagating particles) we now quickly review the incorporation of forces into action principles (which you should have seen in PHGN350).

$$S = \int_{t_i}^{t_f} L(q,\dot{q})\,dt \qquad \text{where} \qquad L = \underbrace{T}_{\text{kinetic term needed even for free particles}} - \overbrace{V}^{\text{interaction term (encodes effects of forces)}}$$

Demand extremality $\delta S = 0$ under variations $q(t) \to q(t) + \delta q(t)$ that vanish at endtimes $\delta q(t_i) = \delta q(t_f) = 0$

$$\delta S = \int_{t_i}^{t_f} \delta L(q,\dot{q})\,dt = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \underbrace{\frac{\partial L}{\partial \dot{q}}\delta\dot{q}} \right) dt$$

I.B.P. $\Rightarrow$ $\displaystyle\int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\,dt = \underbrace{\int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)dt}_{\frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_i}^{t_f} = 0 \text{ since}} - \int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt$

$$\delta S = \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \underbrace{\delta q}_{\text{arbitrary}}\,dt \quad \Rightarrow \quad \underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}_{\text{Euler-Lagrange equations of motion}}$$

How to use: Specify physical system (and generalized coordinates) w/ L
- a) Either set $q(t_i)$ and $q(t_f)$ and then use extremization directly to find $q(t)$.
- b) More commonly set $q(t_i)$ and $\dot{q}(t_i)$ and solve e.o.m. for $q(t)$.

This is all nice and <u>deterministic</u> as is expected for classical systems.

We could generalize this to relativistic particles, but particle creation/annihilation complicates matters.

Instead...

Relativistic Field Theory: $L \to \mathcal{L}(\phi, \partial_\mu \phi) \Rightarrow S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$ $\quad \partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu} \quad x^\mu = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right)}_{\Rightarrow 0 \text{ if } \delta\phi = 0 \text{ asymptotically}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi \right] d^4x$$

$$\Rightarrow \quad \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial_\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}_{\text{Euler-Lagrange e.o.m.}}$$

If: $\mathcal{L} = \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2} \left(\frac{mc}{\hbar} \right)^2 \phi^2 \Rightarrow \partial_\mu \partial^\mu \phi - \left(\frac{mc}{\hbar} \right)^2 = 0$ The Klein-Gordon Equation

If: $\mathcal{L} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \partial_\mu F^{\mu\nu} = 0 = \frac{1}{2}$ Maxwell's Equations

Again, this is all nice and classical, i.e. prescribe $\mathcal{L}$ w/ boundary conditions the solve for (unique) classical field configuration.

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Quantum

We know that in quantum physics the questions (and answers) change. We do not set up a system w/ initial conditions and then ask "What will happen?", but rather "What is the probability of a particular outcome?"

Comparing:

Classical

given L (or H)

set initial • ⇒ predict final •

$q(t_i)$ and $\dot{q}(t_i)$ — solve e.o.m.

Quantum

given L (or H)

set initial • ⇒ predict prob.

$q(t_i)$ or $\dot{q}(t_i)$ — ?

- $q_1(t_f)$
- $q_2(t_f)$
- $q_3(t_f)$

etc.

} All outcomes compatible w/ initial cond. and L (or H)

Typically we start w/ $|q(t_i)\rangle$ and then time evolve w/ $e^{-i\hat{H}(t_f-t_i)}|q(t_i)\rangle$. Then to get prob. for kth result:

$$\underbrace{\langle q_k(t_f)|e^{-i\hat{H}(t_f-t_i)}|q(t_i)\rangle}_{\text{Amplitude to go from } q(t_i) \text{ to } q(t_f)} \Rightarrow |\langle \; | \; \rangle|^2 = \text{prob.}$$

This will be our main focus (from which we can get) abbreviated to $\langle q_f|e^{-i\hat{H}T}|q_i\rangle$

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$\langle q_f | e^{-\frac{i}{\hbar}\hat{H}T} | q_i \rangle = ?$

Break up T into $N+1$ equal time segments:

$= \langle q_f | e^{-\frac{i}{\hbar}\hat{H}\delta t} e^{-\frac{i}{\hbar}\hat{H}\delta t} \cdots e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_i \rangle$

Insert complete set of position states $\int dq\, |q\rangle\langle q| = 1$ between each factor (for $N+1$ factors we need N insertions)

$$= \prod_{j=1}^{N} \int dq_j \underbrace{\langle q_f | e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_N \rangle \langle q_N | e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_{N-2} \rangle \cdots \langle q_1 | e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_i \rangle}_{N+1 \text{ of these}}$$

Consider one term of the form $\langle q_N | e^{-\frac{i}{\hbar}\hat{H}\delta t} | q_{N-1} \rangle$ and $\hat{H} = T + V$ w/ $T = \frac{\hat{p}^2}{2m}$ and $V(\hat{q})$

Then: $\langle q_N | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t - \frac{i}{\hbar}V(\hat{q})\delta t} | q_{N-1} \rangle = \langle q_N | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t - \frac{i}{\hbar}V(q_{N-1})\delta t} | q_{N-1} \rangle$

using $\hat{q}|q_{N-1}\rangle = q_{N-1}|q_{N-1}\rangle$

Now inserting a complete set of momentum states $\int \frac{dp_k}{2\pi} |p_k\rangle\langle p_k| = 1$ we obtain:

$$\int \frac{dp_k}{2\pi} \langle q_N | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\delta t - \frac{i}{\hbar}V(q_{N-1})\delta t} | p_k \rangle \langle p_k | q_{N-1} \rangle = \int \frac{dp}{2\pi} e^{-\frac{i}{\hbar}\frac{p_k^2}{2m}\delta t - \frac{i}{\hbar}V(q_{N-1})\delta t} \langle q_N | p_k \rangle \langle p_k | q_{N-1} \rangle$$

Now use that $\langle q | p_k \rangle = e^{i p_k q}$ $\left(\langle p_k | q \rangle = \langle q | p_k \rangle^* = e^{-i p_k q} \right)$ to obtain:

$$\int \frac{dp_k}{2\pi} e^{-\frac{i}{\hbar}\left[\frac{p_k^2}{2m} + V(q_{N-1})\right]\delta t} e^{\left[\frac{i}{\hbar}p_k q_N - \frac{i}{\hbar}p_k q_{N-1}\right]\delta t} = e^{-\frac{i}{\hbar}V(q_{N-1})\delta t} \int \frac{dp_k}{2\pi} e^{-\frac{i}{\hbar}\frac{p_k^2}{2m}\delta t + \frac{i}{\hbar}(q_N - q_{N-1})p_k \delta t}$$

Using that $\int_{-\infty}^{\infty} dx\, e^{\frac{1}{2}iax^2 + iJx} = \left(\frac{2\pi i}{a}\right)^{1/2} e^{-\frac{iJ^2}{2a}}$ w/ $x = p_k$, $a = \frac{-\delta t}{m\hbar}$, $J = (q_N - q_{N-1})\frac{\delta t}{\hbar}$ we obtain

$$\frac{1}{2\pi} e^{-\frac{i}{\hbar}V(q_{N-1})\delta t} \left(-\frac{2\pi i m \hbar}{\delta t}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\delta t \left(\frac{q_N - q_{N-1}}{\delta t}\right)^2}$$

Recall this is for the $\langle q_N | e^{-i\hat{H}\delta t} | q_{N-1} \rangle$ term.

Going back to the full expression we have:

$$\langle q_f | e^{-\frac{i}{\hbar}HT} | q_i \rangle = \left(-\frac{imh}{2\pi\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \int dq_j\, e^{\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{q_f - q_N}{\delta t}\right)^2 - V(q_N)\right]\delta t} e^{\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{q_N - q_{N-1}}{\delta t}\right)^2 - V(q_{N-1})\right]\delta t} \cdots e^{\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{q_1 - q_i}{\delta t}\right)^2 - V(q_i)\right]\delta t}$$

$$= \left(-\frac{imh}{2\pi\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \int dq_j\, e^{\sum_{k=0}^{N} \frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{q_{k+1} - q_k}{\delta t}\right)^2 - V(q_k)\right]\delta t}$$

Going to the continuum limit $(\delta t \to 0)$ we replace $\frac{q_{k+1} - q_k}{\delta t} = \dot{q}$ and $\delta t \sum_{k=0}^{N} = \int_0^T dt$ and define

$\int Dq(t) \equiv \lim_{N\to\infty} \left(\frac{imh}{2\pi\delta t}\right)^{\frac{N+1}{2}} \prod_{j=1}^{N} \int dq_j$ we obtain:

$$\langle q_f | e^{-\frac{i}{\hbar}HT} | q_i \rangle = \int Dq(t)\, e^{\frac{i}{\hbar}\int_0^T \left[\frac{1}{2}m\dot{q}^2 - V(q)\right]dt} = \int Dq(t)\, e^{\frac{i}{\hbar}\int_0^T L\, dt} = \int Dq(t)\, e^{\frac{i}{\hbar}S}$$

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Again: $\langle q_f | e^{-\frac{i}{\hbar}\hat{H}T} | q_i \rangle = \underbrace{\int Dq(t)\, e^{\frac{i}{\hbar}S}}$

Path integral governed by choice of L (from H) and $q(t_i)$ and $q(t_f)$.

Important points:

Recall that in contrast to the classical case, this is not computing "the" path from $q(t_i)$ to $q(t_f)$ but only the amplitude (hence probability) that if we start at $q(t_i)$ w/ L that we end up at $q(t_f)$.

Also, classically we use S by demanding $\delta S = 0$ which scans over all paths and selects "the" one extremizing S, throwing the other paths out.
In this case we integrate over all paths weighting each one by $e^{\frac{i}{\hbar}S}$ ← different values for different paths

The classical limit is transparent since as $\hbar \to 0$, $\frac{S}{\hbar}$ is large and hence $e^{i\frac{S}{\hbar}}$ oscillates rapidly as we scan over paths. After cancellations the path integral is dominated by the classical path, i.e. the one for which $\delta S = 0$.

This helped Feynman (and others) feel a little bit better about the principle of least action. The question of how a classical particle knows which path to take, is answered by: The particle explores all paths, but only really cares about the one for which $\delta S = 0$.

The path integral approach is immediately generalizable to fields, e.g.

$\int D\phi\, e^{\frac{i}{\hbar}\int \mathcal{L}(\phi, \partial_\mu \phi) d^4x}$ computes the amplitude for evolving to a final field configuration ϕ_f after starting w/ ϕ_i (and specified $\mathcal{L}$).

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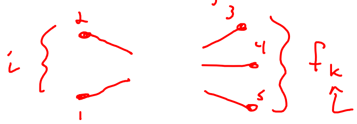
In practice we are often interested in computing transition amplitudes between an initial configuration of particles and a final configuration of particles.



$\uparrow$ different possible outcomes

creation operators vacuum

We can create these particle-like states from the field vacuum w/ $\underbrace{\phi(x_1)\phi(x_2)}_{\text{creation operators}}|0\rangle$

Then $\mathcal{M} = \langle f | i \rangle = \langle 0 | \phi(x_5)\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1) | 0 \rangle \equiv \underbrace{G_5(x_1, x_2, x_3, x_4, x_5)}_{\text{5-point correlation function}}$

$G_2(x_1, x_2)$ is the Green's function

$G_2(x_1, x_2)$ for a free theory is the Feynman propagator

In terms of a path integral $$G_n(x_1, x_2, \ldots, x_n) = \underbrace{\frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n) e^{\frac{i}{\hbar}S[\phi]}}{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S[\phi]}}}$$

This sucks, because it means we have to evaluate a new path integral for each different process (or n-point function)

But... if we can calculate $Z[J] = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}[S[\phi] + \int d^4x\, J(x)\phi(x)]}$ just once when $J(x)$ is a source current for the field $\phi(x)$, then we have:

$$G_n(x_1, x_2 \cdots x_n) = \underbrace{(-i\hbar)^n \frac{1}{Z[0]} \frac{\partial^n Z}{\partial J(x_1)\partial J(x_2)\cdots\partial J(x_n)}}\Bigg|_{J=0}$$

Look, no more integrals... only derivatives

For obvious (if you have studied statistical mechanics) $Z[J]$ is called the "partition function" of the QFT.

We should have expected similar structures from QFT and classical statistical mechanics. Both involve many degrees of freedom and probability distributions arising from working in terms of less than all the information (by choice in stat-mech and by necessity in QFT). The also both use nontrivial weighted sums as the basis for their probability distributions.